ON THE PROJECTION AND MACPHAIL CONSTANTS OF l_n^p SPACES

BY

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ABSTRACT

We prove that the projection and Macphail constants of l_n^p $(1 \leqq p \leqq 2)$ are asymptotically equivalent to $n^{1/2}$ and $n^{-1/2}$ respectively. We also obtain some relations linking certain parameters of general finite dimensional real Banach spaces.

Preliminaries and definitions. Let l_n^p $(1 \leqq p < \infty)$ be the space of real n-tuples $x = (x_1, x_2, \cdots, x_n)$ with the norm $\| x \|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, and let l_n^∞ be the same space with the norm $\| x \|_\infty = \sup \{|x_i|; 1 \leqq i \leqq n\}$.

If X, Y are two isomorphic Banach spaces, we denote the "distance coefficient" between them by $d(X,Y) = \inf \| T \| \, \| T^{-1} \|$, where the infimum is taken over all the bounded linear one-to-one transformations T from X onto Y.

The "projection constant" of X, denoted by $\lambda(X)$, is defined to be the infimum of the numbers λ such that for every Banach space Y containing X as a subspace, there exists a linear projection P from Y onto X with norm not exceeding λ (if there is no such λ, we write $\lambda(X) = \infty$).

The "Macphail constant" of the space X, denoted by $\mu(X)$, is defined as inf $\{(\sup \| \sum_{i \in J} a_i \|) / \sum_{i=1}^m \| a_i \| \}$, where the supremum is taken over all subsets J of $\{1, 2, \cdots, m\}$, and the infimum is taken over all finite sets $\{a_i \in X; \sum_{i=1}^m \|a_i\| > 0\}$.

If f, g are two positive functions defined on the integers, we write $f(n) \sim g(n)$ if $\inf_n (f(n)/g(n)) > 0$ and $\sup_n (f(n)/g(n)) < \infty$. All the spaces considered here will be assumed to be real Banach spaces.

We summarise first the known results concerning the constants which were defined above for l_n^p spaces:

(i) If $1 \leqq p \leqq q \leqq \infty$ and $(p-2)(q-2) \geqq 0$, then $d(l_n^p, l_n^q) = n^{1/p - 1/q}$, [4].

(ii) If $1 \leqq p \leqq 2 \leqq q \leqq \infty$, then $d(l_n^p, l_n^q) \sim \max \{n^{1/p - 1/2}, n^{1/2 - 1/q}\}$, [4].

(iii) If $2 \leqq p \leqq \infty$, then $\lambda(l_n^p) \sim n\mu(l_n^p) \sim n^{1/p}$, [8].

(iv) If $1 \leqq p \leqq 2$, then $\lambda(l_n^p) \leqq (1 + \sqrt{2}) \sqrt{n}$, [4].

(v) $\lambda(l_n^1) \sim \sqrt{n}$ (the exact value was calculated in [3]).

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(vi) $\lambda(l_n^2) \sim \sqrt{n}$ (in [3] an upper bound was found for $\lambda(l_n^2)$, and this bound was later shown to be exact in [8]).

We prove here that $\lambda(l_n^p) \sim \sqrt{n}$ for $1 \leqq p \leqq 2$. This solves a problem raised in [5] and [8].

THEOREM 1. *If* $1 \leqq p \leqq 2$ *then*

$$\lambda(l_n^p) \sim n\mu(l_n^p) \sim \sqrt{n}. \tag{1}$$

In the proof of Theorem 1 we shall use the following result which is of interest in itself.

THEOREM 2. *Let X be an n-dimensional Banach space, then*

$$2n\mu(X) \leqq \lambda(X). \tag{2}$$

Proof. In view of the continuity of $\lambda(X)$ and $\mu(X)$ as a function of X (i.e. if $d(X_m, X) \to 1$ then $\lambda(X_m) \to \lambda(X)$, $\mu(X_m) \to \mu(X)$), it is enough to prove (2) for a polyhedral space Z (i.e. a Minkowsky space whose unit ball is a polytope). It is well known and easily seen that every polyhedral space is isometrically embeddable in a suitable l_m^∞, and thus we may assume that $Z \subseteq l_m^\infty$. It is also well known that $\lambda(Z) = \min\| P \|$, where P ranges over all the linear projections from l_m^∞ onto Z.

Let P be any projection from l_m^∞ onto Z such that $\| P \| = \lambda(Z)$. Let $e_i = (\underbrace{0, \cdots, 0, 1}_{i}, 0, \cdots, 0)$ $(1 \leqq i \leqq m)$ be the natural basis of l_m^∞, and put $e_{m+i} = -e_i (1 \leqq i \leqq m)$.

Obviously for every subset $J \subseteq \{1, 2, \cdots, 2m\}$

$$\left\| \sum_{i \in J} Pe_i \right\| \leqq \| P \| = \lambda(Z). \tag{3}$$

On the other hand, if $Pe_i = \sum_{j=1}^m \alpha_{i,j} e_j$, then trace $P = \sum_{i=1}^m \alpha_{i,i} = n$ (VI.9.28, [1]), and thus

$$\sum_{i=1}^{2m} \| Pe_i \| = 2 \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \alpha_{i,j} e_j \right\| \geqq 2 \sum_{i=1}^{m} |\alpha_{i,i}| \geqq 2n. \tag{4}$$

By combining (3) and (4) we get

$$\mu(Z) \leqq \sup_J \left\| \sum_{i \in J} Pe_i \right\| \Big/ \sum_{i=1}^{2m} \| Pe_i \| \leqq \lambda(Z)/2n.$$

REMARK. Equation (2) cannot be improved in general since $\lambda(l_n^\infty) = 2n\,\mu(l_n^\infty) = 1$ [8].

We shall need also the following technical lemma whose proof is very similar to that of Theorem 1 (ii) of [8]. We use the following notations: Let X be an n-dimensional Banach space. We denote by $\| \ \|_X$ and $\| \ \|_{X^*}$ the norms in X and X^* respectively. We fix in X a coordinate system. The Euclidean norm with respect to these coordinates (in X and X^*) will be denoted by $\| \ \|_2$. By σ_n and ω_n we denote the surface area and volume respectively of $B_n = \{x; \|x\|_2 \leqq 1\}$, and by $d\sigma_n$ the element of area on $S_n = \{x; \|x\|_2 = 1\}$.

LEMMA 1. *Let X be an n-dimensional Banach space, then*

$$\mu(X) \cdot \int_{S_n} \|y\|_{X^*} d\sigma_n \cdot \sup_{\|y\|_2 = 1} \|y\|_X \geqq \omega_{n-1}. \tag{5}$$

Proof. Let $A = \{a_1, a_2, \cdots, a_m\}$ be any finite subset of X, and let $B = \{ \sum_{i=1}^m \lambda_i a_i;\ |\lambda_i| \leqq 1,\ 1 \leqq i \leqq m\}$. Clearly B is a polytope in X, all whose extreme points are of the form $\sum_{i=1}^m \varepsilon_i a_i$ with $\varepsilon_i = \pm 1$. Hence

$$\begin{aligned} \gamma &= \max\{\|x\|_X; x \in B\} = \max_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^m \varepsilon_i a_i \Big\|_X \\ &\leqq 2 \max_J \Big\| \sum_{i \in J} a_i \Big\|_X, \end{aligned} \tag{6}$$

where J ranges over the subsets of $\{1, 2, \cdots, m\}$. Clearly

$$\begin{aligned} \gamma &= \sup\{|(x,y)| / \|y\|_{X^*}; x \in B, y \in S_n\} \\ &= \sup\Big\{ \sum_{i=1}^m |(a_i, y)| / \|y\|_{X^*}; y \in S_n \Big\}. \end{aligned}$$

Hence for every $y \in S_n$, $\gamma \|y\|_{X^*} \geqq \sum_{i=1}^m |(a_i, y)|$. By integrating over the unit sphere S_n, we get

$$\begin{aligned} \gamma \int_{S_n} \|y\|_{X^*} d\sigma_n &\geqq \sum_{i=1}^m \int_{S_n} |(a_i, y)| \, d\sigma_n = \sum_{i=1}^m 2\omega_{n-1} \|a_i\|_2 \\ &\geqq 2\omega_{n-1} \sum_{i=1}^m \|a_i\|_X \Big/ \sup_{\|x\|_2 = 1} \|x\|_X, \end{aligned}$$

and thus by (6),

$$\sup_{\|x\|_2 = 1} \|x\|_X \cdot \int_{S_n} \|y\|_{X^*} d\sigma_n \cdot \Big(\sup_J \Big\| \sum_{i \in J} a_i \Big\|_X \Big/ \sum_{i=1}^m \|a_i\|_X \Big) \geqq \omega_{n-1}.$$

Since A was arbitrary the proof is concluded.

We may now prove Theorem 1 for the case $1 < p \leqq 2$. We shall see later that (1) is an immediate consequence of (13), however the following proof is simpler and more direct.

Proof of Theorem 1 for $1 < p \leqq 2$: Applying Theorem 2 and Result (iv) we obtain

$$2n\mu(l_n^p) \leqq \lambda(l_n^p) \leqq (1 + \sqrt{2})\sqrt{n}. \tag{7}$$

We use (5) in order to find a lower bound for $\mu(l_n^p)$. We need thus an estimate for $\sigma_n^{-1}\int_{S_n}\|x\|_q d\sigma_n$, where $q = p/(p-1)$. By Hölder's inequality

$$(8)\qquad \sigma_n^{-1}\int_{S_n}\|x\|_q d\sigma_n \leqq \left(\sigma_n^{-1}\int_{S_n}\|x\|_q^q d\sigma_n\right)^{1/q} = \left(n\sigma_n^{-1}\int_{S_n}|x_1|^q d\sigma_n\right)^{1/q}.$$

The points of S_n may be defined by spherical coordinates $x_1 = \sin\theta_1$, $x_2 = \sin\theta_2\cos\theta_1, \cdots, x_{n-1} = \sin\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_1$, $x_n = \cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_1$, where $-\pi \leqq \theta_{n-1} \leqq \pi$, $-\pi/2 \leqq \theta_k \leqq \pi/2$ $(1 \leqq k \leqq n-2)$, and

$$d\sigma_n = \cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos\theta_{n-2}d\theta_1\cdots d\theta_{n-1}.$$

Then continuing (8) and substituting $t = \cos^2\theta_1$, we have

$$(9)\qquad n\sigma_n^{-1}\int_{S_n}|x_1|^q d\sigma_n = 2n\sigma_{n-1}\sigma_n^{-1}\int_0^{\pi/2}\sin^q\theta_1\cos^{n-2}\theta_1 d\theta_1$$

$$= n\sigma_{n-1}\sigma_n^{-1}\int_0^1 t^{(n-3)/2}(1-t)^{(q-1)/2}dt$$

$$= n\sigma_{n-1}\sigma_n^{-1}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Big/\Gamma\left(\frac{n+q}{2}\right).$$

Since

$$\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Big/\Gamma\left(\frac{n+q}{2}\right) \sim n^{-q/2-1/2},$$

and $n\sigma_{n-1}\sigma_n^{-1} \sim n^{3/2}$, it follows that $n\sigma_n^{-1}\int_{S_n}|x_1|^q d\sigma_n \sim n^{1-q/2}$, and hence by (8)

$$(10)\qquad \sigma_n^{-1}\int_{S_n}\|x\|_q d\sigma_n \leqq c_1 n^{1/q-1/2}, \text{ where } c_1 = c_1(p) > 0.$$

Also $\sup_{\|x\|_2=1}\|x\|_p = n^{1/p-1/2}$ and $\omega_{n-1}/\sigma_n \sim n^{-1/2}$, and this together with (10) when applied to (5) yields the required inequality $\mu(l_n^p) \geqq cn^{-1/2}$, where $c = c(p) > 0$ is a constant.

Remark. In the case of l_n^1, Theorem 2 and Result (v) give us

$$2n\mu(l_n^1) \leqq \lambda(l_n^1) \sim n^{1/2}.$$

The use of (5) however requires the estimation of $\sigma_n^{-1}\int_{S_n}\|x\|_\infty d\sigma_n$. By using similar methods to those employed in the proof of Theorem 3(*B*) [2], we find that $\sigma_n^{-1}\int_{S_n}\|x\|_\infty d\sigma_n \sim \sqrt{\log n/n}$. Consequently $\mu(l_n^1) \geqq c/\sqrt{n\log n}$, where $c > 0$ is a constant. However, this is not the best possible asymptotic estimate for $\mu(l_n^1)$, and Corollary 1 will yield the sharper result stated in (1).

Theorem 3. *Let X and Y be n-dimensional Banach spaces, then*

(i) $\mu(X) \leqq \mu(Y)d(X,Y)$

(ii) $\mu(X^*) \leqq 2\mu(X)d(X,l_n^2)$.

Proof. (i) Let $T: Y \to X$ be a linear transformation such that $\| Ty \| \leqq \| y \|$ for every $y \in Y$, and $\| T^{-1} \| = d(X, Y)$. Then the result follows directly from the definition of μ and the inequality $\| Ty \| \leqq \| y \| \leqq d(X, Y) \| Ty \|$ for every $y \in Y$.

(ii) We retain the notations of Lemma 1, and choose a coordinate system in X, so that $K \subseteq B_n \subseteq d(X, l_n^2)K$, where K is the unit ball of X. Let

$$F = \left\{ (\mu(X^*))^{-1} \sum_{i=1}^{m} |x_i^*|;\ \sum_{i=1}^{m} \| x_i^* \| = 1 \right\}.$$

It is easily verified that F is a convex subset of $C(K)$ (the Banach space of continuous functions on K) which is disjoint from the set $G = \{ f \in C(K);\ \sup_{x \in K} f(x) < 1 \}$.

Since G is a convex open set which contains the open unit ball of $C(K)$ and the negative functions in $C(K)$, it follows from the separation theorems and the Riesz representation theorem, that there exists a probability measure v on K such that $\int f dv \geqq 1$ for every $f \in F$. For any $0 \neq x^* \in X^*$, the function $f = |x^*| / \mu(X^*) \| x^* \|$ belongs to F, hence

$$\mu(X^*) \| x^* \| \leqq \int |x^*(x)| dv(x). \tag{11}$$

Integrating (11) over S_n we get

$$\begin{aligned} \mu(X^*) \int_{x^* \in S_n} \| x^* \| d\sigma_n &\leqq \int_{x^* \in S_n} \left(\int |x^*(x)| dv(x) \right) d\sigma_n \\ &= \int \left(\int_{x^* \in S_n} |x^*(x)| d\sigma_n \right) dv(x) = 2\omega_{n-1} \int \| x \|_2 dv(x) \qquad (12) \\ &\leqq 2\omega_{n-1} \int 1 dv(x) = 2\omega_{n-1}, \end{aligned}$$

and thus by (5) and (12) $\mu(X^*) \leqq 2\mu(X) d(X, l_n^2)$.

THEOREM 4. *Let X be an n-dimensional Banach space, then*

$$\mu(X) d(X, l_n^2) d(X, l_n^1) \geqq 1/2K_G, \tag{13}$$

where K_G is the universal Grothendieck constant

$$\left(\frac{\pi}{2} \leqq K_G \leqq \sinh \frac{\pi}{2} \right).$$

Proof. The proof is essentially contained in Theorem 4.1 [7], however for the sake of completeness we rephrase it to fit our definitions. We shall use the following result due to Grothendieck (see e.g. Theorem 2.1 [7]):

Let $\{a_{i,j}\}_{i,j=1,2,\cdots,m}$ be a real valued matrix, and let M be a positive number such that $| \sum_{i,j=1}^{m} a_{i,j} t_i s_j | \leqq M$ for every real $\{t_i\}_{i=1}^{m}$ and $\{s_j\}_{j=1}^{m}$ satisfying

$|t_i| \leqq 1$ and $|s_j| \leqq 1$. Then for arbitrary vectors $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^m$ in a real inner product space H

$$\left| \sum_{i,j=1}^{m} a_{ij}(x_i, y_j) \right| \leqq K_G M \sup_i \| x_i \| \sup_j \| y_j \|. \tag{14}$$

In particular, if for a given set $\{x_i\}_{i=1}^m$ we choose the set $\{y_j\}_{j=1}^m$ such that $\| y_j \| = 1$ and $(\sum_{i=1}^m a_{i,j}x_i, y_j) = \| \sum_{i=1}^m a_{i,j}x_i \|$, we obtain from (14)

$$\sum_{j=1}^{m} \| \sum_{i=1}^{m} a_{i,j}x_i \| \leqq K_G M \sup_i \| x_i \|. \tag{15}$$

We turn to the proof of Theorem 4. There exists a transformation $S: l_n^1 \to X$ such that $\| S \| = 1$ and $\| S^{-1} \| = d(X, l_n^1)$. Let $\{x_i\}_{i=1}^m \subset X$ be an arbitrary set, put $y_i = S^{-1}x_i$. Let $\{e_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ be the natural bases in l_n^1 and $(l_n^1)^* = l_n^\infty$ respectively, and let $y_i = \sum_{j=1}^n a_{i,j}e_j$. Let $\{s_j\}_{j=1}^n$, $\{t_i\}_{i=1}^m$ be any real numbers with absolute values $\leqq 1$, and put $y^* = \sum_{j=1}^n s_j f_j$. Then

$$\begin{aligned} \left| \sum_{i,j} a_{i,j}t_i s_j \right| &\leqq \sum_{i=1}^{m} |t_i| \left| \sum_{j=1}^{n} a_{i,j}s_j \right| \\ &\leqq \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{i,j}s_j \right| = \sum_{i=1}^{m} |y^*(y_i)| \\ &\leqq \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^{m} \varepsilon_i y_i \| \\ &\leqq \|S^{-1}\| \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^{m} \varepsilon_i x_i \|. \end{aligned} \tag{16}$$

There exists a transformation $T: X \to l_n^2$ such that $\| T^{-1} \| = 1$ and $\| T \| = d(X, l_n^2)$. Then using (15) we obtain

$$\begin{aligned} \sum_{i=1}^{m} \| x_i \| &\leqq \sum_{i=1}^{m} \| Tx_i \| = \sum_{i=1}^{m} \| TSy_i \| = \sum_{i=1}^{m} \| \sum_{j=1}^{n} a_{i,j}TSe_j \| \\ &\leqq K_G \| S^{-1} \| \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^{m} \varepsilon_i x_i \| \, \| T \| \, \| S \| \\ &\leqq 2K_G d(X, l_n^1) d(X, l_n^2) \sup_J \| \sum_{i \in J} x_i \|, \end{aligned} \tag{17}$$

where J denotes a subset of $\{1, 2, \cdots, m\}$, and this proves inequality (13).

COROLLARY 1. *For* $1 \leqq p \leqq 2$,

$$(1 + \sqrt{2}) \sqrt{n} \geqq \lambda(l_n^p) \geqq 2n\mu(l_n^p) \geqq \sqrt{n} / K_G.$$

Proof. The upper bound is Result (iv). For the lower bound, take $X = l_n^p$ in Theorem 4 and use Result (i) and Theorem 2.

Corollary 1 proves Theorem 1. It follows also that (13) is asymptotically exact for every l_n^p $(1 \leqq p \leqq \infty)$.

COROLLARY 2. *Let X be an n-dimensional subspace of l^1, then*

$$\lambda(X) \geqq K_G^{-1/2} n^{1/4} .$$

Proof. Let $T: X \to l_n^2$ be a transformation for which $\| T \| \; \| T^{-1} \| = d(X, l_n^2)$ and let P be a projection of l^1 on X. Now, $TP: l^1 \to l_n^2$, and as in the proof of Theorem 4, we have for any subset $\{x_i\}_{i=1}^m \subset l^1$

$$\| \sum_{i=1}^m TPx_i \| \leqq K_G \| TP \| \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^m \varepsilon_i x_i \| \tag{18}$$

(this is essentially a result of Theorem 4.1, [7]). Taking in particular $x_i \in X$ in (18), we get

$$\| \sum_{i=1}^m x_i \| \leqq \sum_{i=1}^m \| T^{-1} \| \; \| Tx_i \| \leqq K_G d(X, l_n^2) \| P \| \sup_{\varepsilon_i = \pm 1} \| \sum_{i=1}^m \varepsilon_i x_i \|$$

$$\leqq 2K_G d(X, l_n^2) \| P \| \sup_J \| \sum_{i \in J} x_i \| ,$$

where J ranges over the subsets of $\{1, 2, \cdots, m\}$. By the definition of $\mu(X)$

$$\mu(X) \| P \| d(X, l_n^2) \geqq 1/2K_G. \tag{19}$$

Consequently

$$\mu(X)\lambda(X)d(X, l_n^2) \geqq 1/2K_G. \tag{20}$$

Applying (2) and the inequality $d(X, l_n^2) \leqq \sqrt{n}$ [6] in (20), we obtain the required result.

COROLLARY 3. *Let X be an n-dimensional Banach space, then*

$$\lambda(X)\lambda(X^*) \geqq K_G^{-2/3} n^{1/3}. \tag{21}$$

Proof. Due to the continuity of $\lambda(X)$, we may assume, as in the proof of Theorem 2, that X is a subspace of a suitable l_m^∞. Let $P: l_m^\infty \to X$ be a projection such that $\lambda(X) = \| P \|$, and let $I: X \to l_m^\infty$ be the identity on X. Then P^*I^* is a projection of $(l_m^\infty)^* = l_m^1$ onto P^*X^*, and it follows from (19) that

$$\mu(P^*X^*) \| P^*I^* \| d(P^*X^*, l_n^2) \geqq 1/2K_G. \tag{22}$$

But by Theorem 3(i),

$$\mu(P^*X^*) \leqq \mu(X^*) d(P^*X^*, X^*) \leqq \mu(X^*) \| P \| \leqq \lambda(X^*)\lambda(X)/2n,$$

and since $d(P^*X^*, l_n^2) \leqq \sqrt{n}$ and $\| P^*I^* \| \leqq \| P \| = \lambda(X)$, we obtain from (22): $\lambda(X^*)(\lambda(X))^2 \geqq \sqrt{n}/K_G$.

Similarly, $\lambda(X)(\lambda(X^*))^2 \geqq \sqrt{n}/K_G$, and (21) follows by multiplying both inequalities.

REMARK. An upper bound for $\lambda(X)\lambda(X^*)$ may be found in [5], where it was shown that $d(X, l_n^\infty)d(X, l_n^1) \leqq n$ if X is a real n-dimensional symmetric Banach space. The corresponding inequality for a non-symmetric space involves the asymmetry constants of the space. Since $\lambda(X) \leqq d(X, l_n^\infty) = d(X^*, l_n^1)$ if X is n-dimensional, it follows that if X is symmetric as well, then

$$\lambda(X)\lambda(X^*) \leqq d(X, l_n^\infty)d(X, l_n^1) \leqq n.$$

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